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Charged dilaton, energy, momentum and angular-momentum in teleparallel theory equivalent to general relativity

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Abstract. We apply the energy-momentum tensor to calculate energy, momentum and angular-momentum of two different tetrad fields. This tensor is coordinate independent of the gravitational field established in the Hamiltonian structure of the teleparallel equivalent of general relativity (TEGR). The spacetime of these tetrad fields is the charged dilaton. Our results show that the energy associated with one of these tetrad fields is consistent, while the other one does not show this consistency. Therefore, we use the regularized expression of the gravitational energy-momentum tensor of the TEGR. We investigate the energy within the external event horizon using the definition of the gravitational energy-momentum.

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1 Introduction

Quantum mechanics and general relativity (GR) are two very successful and well validated theories within their own domains. The main problem is to unify them into a single consistent theory. One of the most promising models of unification is string theory. String theory is classified into two classes, which are closed and the open strings. Gravity is described by the first class, while matter is described by the second one. In case of non-perturbative string theory, there are extended objects known as D-branes. These objects are surfaces where open strings must begin and finish; they provide an alternative approach to the one of Kaluza–Klein [1, 2]. In this latter approach the matter penetrates the extra dimensions, leading to strong constraints from collider physics.

Nowadays, there is a growing body of literature on the gravitational field of string matter coupled to an electromagnetically charged dilaton field. Black hole solutions in dilaton gravity were first analyzed by Gibbons and Maeda [3]. Garfinkle et al. [4] have obtained a family of solutions representing static, spherically symmetric charged black holes. Kallosh and Peet [5, 6] in the context of supersymmetric theories investigated these solutions. When the dilaton acquires a mass Gregory and Harvey [7] modified the dilaton black holes. A static spherically symmetric metric around a source coupled to a massless dilaton with both electric and magnetic charges has been investigated by Agnese and Camera [8].

Among various attempts to overcome the problems of quantization and the existence of singular solution in Einstein's GR, gauge theories of gravity are of special interest, as they are based on the concept of gauge symmetry, which has been very successful in the foundation of other fundamental interactions. The importance of Poincaré symmetry in particle physics leads one to consider Poincaré gauge theory (PGT) as a natural framework for the description of the gravitational phenomena [9–19]. Basic gravitational variables in PGT are the tetrad field ${e^a}_\mu$ and the Lorentz connection ${A^{ab}}_\mu$. These variables are associated to the translation and Lorentz subgroups of the Poincaré group. The gauge fields are coupled to the energy-momentum and spin of the matter fields, and their field strengths are geometrically identified with the torsion and the curvature.

The general geometric arena of PGT, the Riemann–Cartan space U_4, may be a priori restricted by imposing certain conditions on the curvature and the torsion. Thus, Einstein's GR is defined in Riemann space V_4, which is obtained from U_4 by the requirement of vanishing torsion. Another interesting limit of PGT is the *teleparallel or Weitzenböck* geometry T_4. The vanishing of the curvature means that parallel transport is path independent. The teleparallel geometry is, in a sense, complementary to Riemannian geometry: curvature vanishes, and torsion remains to characterize parallel transport. For the physical interpretation of teleparallel geometry there is a one-parameter family of teleparallel Lagrangians, which is empirically equivalent to GR [18, 20, 21]. If the parameter value $B = 1/2$ the Lagrangian of the theory coincides, modulo a four-divergence, with the Einstein–Hilbert Lagrangian, and it defines TEGR.

The search for a consistent expression for the gravitating energy and angular-momentum of a self-gravitating

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distribution of matter is undoubtedly a long-standing problem in GR. It is believed that the energy of the gravitational field is not localizable, i.e., defined in a finite region of the space. The gravitational field does not possess the proper definition of an energy-momentum tensor. It is usual to define some energy-momentum and angular-momentum [22, 23] that are pseudo-tensors and depend on the second derivative of the metric tensor. These quantities can be annulled by an adequate coordinate transformation. Bergmann [22] and Landau–Lifshitz [23] justify that the energy and angular-momentum are consistent with Einstein's principle of equivalence. According to this principle "any space-time region, infinitesimal or not, is flat if and only if the Riemann–Christoffel tensor vanishes in this region." In such a flat space-time, the energy of the gravitational field is null. Therefore, it is only possible to define the energy of the gravitational field in the whole space-time region and not only in a small region. Einstein GR can also be reformulated in the context of teleparallel geometry [24–42]. In this geometry the dynamical field corresponds to the orthonormal tetrad field ${e^a}_\mu$ (a, μ are SO(3,1) and space-time indices, respectively). Teleparallel geometry is a suitable framework to address the notions of energy, momentum and angular-momentum of any space-time that admits a $3+1$ foliation [43]. Therefore, we consider TEGR in this work.

In order to calculate the energy and angular-momentum we use the Hamiltonian that is formulated for *an arbitrary teleparallel theories* using Schwinger's time gauge [44–57]. In this formulation it is shown that TEGR is the *only viable consistent teleparallel theory of gravity*. Maluf and Rocha [58] established a theory in which *Schwinger's time gauge has not been incorporated in the geometry of absolute parallelism*. In this formulation, the definition of gravitational angular-momentum arises by suitably interpreting the integral form of the constraint equation $\Gamma^{ab}=0$. This definition has been successfully applied to the gravitational field of a thin, slowly rotating mass shell [59] and for the three-dimensional BTZ black hole [60, 61].

Definitions for the gravitational energy in the context of the TEGR have already been proposed in the literature [31–37, 45–50]. Expressions for the gravitational energy arise from the surface term of the total Hamiltonian, given in [62, 63]. These expressions are equivalent to the integral form of the total divergences of the Hamiltonian density developed by Maluf et al. [58]. These expressions yield the same value for the total energy of the gravitational field. However, since these expressions contain the lapse function in the integrand, none of them are suitable to the calculation of the irreducible mass of the Kerr black hole. This is because the lapse function vanishes on the external event horizon of the black hole [45–50]. The energy expressions [62, 63] are neither to be applied to a finite surface integration nor do they yield the total energy of the space-time [45–50]. A good energy-momentum expression for gravitating systems should satisfy a variety of requirements; to give the standard values of the total quantities for asymptotically flat space, to reduce to the material energy-momentum in the proper limit and to be positive [64, 65]. No entirely satisfactory expression has yet been identified. For more details on the topic of the quasi-local approach a review article is referred to [66].

To calculate the energy and momentum, the definition of energy-momentum, i.e., P^a, is given, which is invariant under global SO(3,1) transformations. It has been argued elsewhere [67, 68] that it makes sense for P^a to have a dependence on the frame. The energy-momentum in classical theories of particles and fields does not depend on the frame, and it has been asserted that such a dependence is a natural property of the gravitational energy-momentum. It is assumed that a set of tetrads fields is adapted to an observer in the space-time determined by the metric tensor $g_{\mu\nu}$.

We investigate the irreducible mass $M_{\rm irr}$ of the dilaton black hole. This $M_{\rm irr}$ is the total mass of the black hole at the final stage of Penrose's process of energy extraction, considering that the maximum possible energy is extracted. $M_{\rm irr}$ is also related to the energy contained within the external event horizon $E(r_+)$ of the black hole (the surface of the constant radius $r=r_+$ defines the external event horizon). Every expression for a local or quasi-local gravitational energy must necessarily yield the value of $E(r_+)$ in close agreement with $2M_{\rm irr}$, since we know beforehand the value of $M_{\rm irr}$ as a function of the initial angular-momentum of the black hole [69]. The evolution of $2M_{\rm irr}$ is a crucial test for any expression of the gravitational energy. $E(r_+)$ has been obtained by means of various energy expressions [70]. The gravitational energy used in this article is the only one that yields a satisfactory value for $E(r_+)$ and that arises in the framework of the Hamiltonian formulation of the gravitational field.

The main aim of the present work is to reformulate the solution given by Garfinkle et al. [4] within the framework of TEGR and then compute energy, momentum and angular-momentum using the energy-momentum tensor. In Sect. 2 we briefly review TEGR theory for the gravitational, electromagnetic and dilaton cases and then we derive the equations of motion. A summary of the derivation of energy and angular-momentum is given in Sect. 3. In Sect. 4, we study the two tetrad fields and then calculate the energy and angular-momentum. To calculate the energy associated with the second tetrad field we use the regularized expression for the gravitational energy-momentum in Sect. 5. The final section is devoted to a discussion and our conclusions.

2 TEGR for the gravitation, electromagnetic and dilaton cases

In a space-time with absolute parallelism the parallel vector fields e_a^μ define the nonsymmetric affine connection

$$\Gamma^\lambda{}_{\mu\nu} \stackrel{\rm def.}{=} {e_a}^\lambda {e^a}_{\mu,\nu}\,, \tag{1}$$

where $e_{a\mu,\nu} = \partial_\nu e_{a\mu}$.[1] The curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$, given by (1), is identically vanishing. The metric tensor $g_{\mu\nu}$ is defined by

$$g_{\mu\nu} \stackrel{\text{def.}}{=} \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{2}$$

with $\eta_{ab} = (-1, +1, +1, +1)$ the metric of Minkowski space-time.

The Lagrangian density for the gravitational field in TEGR, in the presence of matter fields, is given by[2] [45–50]

$$\begin{aligned}\mathcal{L}_G = eL_G &= -\frac{e}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^a T_a\right) - L_\text{m} \\ &= -\frac{e}{16\pi}\Sigma^{abc}T_{abc} - L_\text{m} \,,\end{aligned} \tag{3}$$

where $e = \det(e^a{}_\mu)$. The tensor Σ^{abc} is defined by

$$\Sigma^{abc} \stackrel{\text{def.}}{=} \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c) \,. \tag{4}$$

T^{abc} and T^a are the torsion tensor and the basic vector field defined by

$$\begin{aligned}T^a{}_{\mu\nu} &\stackrel{\text{def.}}{=} e^a{}_\lambda T^\lambda{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \,, \\ T^a{}_{bc} &\stackrel{\text{def.}}{=} e_b{}^\mu e_c{}^\nu T^a{}_{\mu\nu} \,, \quad T^a \stackrel{\text{def.}}{=} T^b{}_b{}^a \,.\end{aligned} \tag{5}$$

The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term [45–50]. L_m represents the Lagrangian density for matter fields.

The electromagnetic Lagrangian density $L_\text{e.m.}$ is [72]

$$\mathcal{L}_\text{e.m.} = eL_\text{e.m.} = ee^{-2\xi} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \,, \tag{6}$$

with $F_{\mu\nu}$ being the Maxwell field associated with a U(1) subgroup of $E_8 \times E_8$ and defined by[3] $F_{\mu\nu} \stackrel{\text{def.}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

Finally the dilaton Lagrangian density L_D is [4]

$$\mathcal{L}_D = eL_D = 2e(\nabla\xi)^2 \,, \tag{7}$$

with ξ being the dilaton.

The gravitational, electromagnetic and dilaton field equations for the system described by $L_G + L_\text{e.m.} + L_D$ are the following:

$$\begin{aligned}e_{a\lambda} e_{b\mu} \partial_\nu (e\Sigma^{b\lambda\nu}) - e\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}\right) &= \frac{1}{2}\kappa e T_{a\mu} \,, \\ \nabla_\mu (e^{-2\xi} F^{\mu\nu}) &= 0 \,, \\ \nabla^2 \xi + \frac{1}{2} e^{-2\xi} F^2 &= 0 \,,\end{aligned} \tag{8}$$

where

$$\begin{aligned}T_{\mu\nu} = 2\Big\{ &\nabla_\mu\xi\nabla_\nu\xi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho\xi\nabla_\sigma\xi \\ &+ e^{-2\xi}\left(g_{\nu\sigma} F_{\mu\rho} F^{\sigma\rho} - \frac{1}{4} g_{\mu\nu} F^2\right)\Big\} \,.\end{aligned}$$

It is possible to prove by explicit calculations that the left hand side of the symmetric field equation (8) is exactly given by [45–50]

$$\frac{e}{2}\left[R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e)\right] .$$

The axial-vector part of the torsion tensor a_μ is defined by

$$a_\mu \stackrel{\text{def.}}{=} \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} = \frac{1}{3} \epsilon_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma} \,, \tag{9}$$

where

$$\epsilon_{\mu\nu\rho\sigma} \stackrel{\text{def.}}{=} \sqrt{-g}\, \delta_{\mu\nu\rho\sigma} \,,$$

and $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$.

3 Energy, momentum and angular-momentum

In the context of Einstein's GR, rotational phenomena are certainly not completely understood issues. The prominent manifestation of a purely relativistic rotation effect is the dragging of inertial frames. If the angular-momentum of the gravitational field of an isolated system has a meaningful notion, then it is reasonable to expect the latter to be somehow related to the rotational motion of the physical sources.

The angular-momentum of the gravitational field has been addressed in the literature by means of various approaches. The oldest approach is based on pseudotensors [22, 23], out of which angular-momentum superpotentials are constructed. An alternative approach assumes the existence of certain Killing vector fields that allow for the construction of conserved integral quantities [73–75]. Finally, the gravitational angular-momentum can also be considered in the context of Poincaré gauge theories of gravity [76–79], either in the Lagrangian or in the Hamiltonian formulation. In the latter case it is required that the generators of spatial rotations at infinity have well defined functional derivatives. From this requirement a certain surface integral arises, whose value is interpreted as the gravitational angular-momentum.

The Hamiltonian formulation of TEGR is obtained by establishing the phase space variables. The Lagrangian density does not contain the time derivative of the tetrad component, e_{a0}. Therefore, this quantity will arise as a Lagrange multiplier [80]. The momentum canonically conjugated to e_{ai} is given by $\Pi^{ai} = \delta L / \delta \dot{e}_{ai}$. The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form $L = p\dot{q} - H$, in terms of e_{ai}, Π^{ai} and the

[1] Space-time indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b \ldots$ run from 0 to 3. Time and space indices are indicated by $\mu = 0, i$, and $a = (0), (i)$.

[2] Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

[3] Heaviside–Lorentz rationalized units will be used throughout this paper.

Lagrange multipliers. The Legendre transformation can be successfully carried out and the final form of the Hamiltonian density has the form [43]

$$H = e_{a0}C^a + \alpha_{ik}\Gamma^{ik} + \beta_k \Gamma^k \,, \tag{10}$$

plus a surface term. Here α_{ik} and β_k are Lagrange multipliers that are identified as

$$\alpha_{ik} = \frac{1}{2}(T_{i0k} + T_{k0i}) \,, \quad \beta_k = T_{00k} \,, \tag{11}$$

and C^a, Γ^{ik} and Γ^k are first class constraints. The Poisson brackets between any two field quantities F and G are given by

$$\{F, G\} = \int \mathrm{d}^3x \left(\frac{\delta F}{\delta e_{ai}(x)} \frac{\delta G}{\delta \Pi^{ai}(x)} - \frac{\delta F}{\delta \Pi^{ai}(x)} \frac{\delta G}{\delta e_{ai}(x)} \right) . \tag{12}$$

We recall that the Poisson brackets $\{\Gamma^{ij}(x), \Gamma^{kl}(x)\}$ reproduce the angular-momentum algebra [45–50].

The constraint C^a is written as $C^a = -\partial_i \Pi^{ai} + h^a$, where h^a is an intricate expression of the field variables. The integral form of the constraint equation $C^a = 0$ motivates the definition of the gravitational energy-momentum four-vector P^a [45–50] by

$$P^a = -\int_V \mathrm{d}^3x \partial_i \Pi^{ai} \,, \tag{13}$$

where V is an arbitrary volume of the three-dimensional space. In the configuration space we have

$$\Pi^{ai} = -4\kappa\sqrt{-g}\Sigma^{a0i} \,, \tag{14}$$

with

$$\partial_\nu \left(\sqrt{-g} \Sigma^{a\lambda\nu} \right) = \frac{1}{4\kappa}\sqrt{-g} e^a{}_\mu (t^{\lambda\mu} + T^{\lambda\mu}) \,,$$

where

$$t^{\lambda\mu} = \kappa \left(4\Sigma^{bc\lambda} T_{bc}^{\mu} - g^{\lambda\mu} \Sigma^{bcd} T_{bcd} \right) .$$

The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature. By making $\lambda = 0$ in (14) and identifying Π^{ai} on the left side of the latter, the integral form of (13) is written as

$$P^a = \int_V \mathrm{d}^3x \sqrt{-g} e^a{}_\mu (t^{0\mu} + T^{0\mu}) \,. \tag{15}$$

Equation (15) suggests that P^a is now understood as the gravitational energy-momentum [45–50]. The spatial component $P^{(i)}$ forms a total three-momentum, while the temporal component $P^{(0)}$ is the total energy [23].

It is possible to rewrite the Hamiltonian density of (10) in the equivalent form [59]

$$H = e_{a0}C^a + \frac{1}{2}\lambda_{ab}\Gamma^{ab} \,, \tag{16}$$

where $\lambda_{ab} = -\lambda_{ba}$ are the Lagrangian multipliers that are identified by $\lambda_{ik} = \alpha_{ik}$ and $\lambda_{0k} = -\lambda_{k0} = \beta_k$. The constraints $\Gamma^{ab} = -\Gamma^{ba}$ [43] embody both constraints Γ^{ik} and Γ^k by means of the relation

$$\Gamma^{ik} = e_a{}^i e_b{}^k \Gamma^{ab} \,, \quad \Gamma^k \equiv \Gamma^{0k} = e_a{}^0 e_b{}^k \Gamma^{ab} \,. \tag{17}$$

The constraint Γ^{ab} can be read as

$$\Gamma^{ab} = M^{ab} + 4\kappa\sqrt{-g} e_{(0)}{}^0 (\Sigma^{a(0)b} - \Sigma^{b(0)a}) \,. \tag{18}$$

In similarity to the definition of P^a, the integral form of the constraint equation $\Gamma^{ab} = 0$ motivates the new definition of the space-time angular-momentum. The equation $\Gamma^{ab} = 0$ implies

$$M^{ab} = -4\kappa\sqrt{-g} e_c{}^0 (\Sigma^{acb} - \Sigma^{bca}) \,. \tag{19}$$

Maluf et al. [45–50, 59] defined

$$L^{ab} = \int_V \mathrm{d}^3x e_\mu{}^a e_\nu{}^b M^{\mu\nu} \tag{20}$$

as the four-angular-momentum of the gravitational field for an arbitrary volume V of the three-dimensional space. In Einstein–Cartan type theories there also appear constraints that satisfy the Poisson brackets as given by (12). However, such constraints arise in the form $\Pi^{[ij]} = 0$, and so a definition similar to (20), i.e., interpreting the constraint equation as an equation for the angular-momentum of the field, *is not possible*. Definition (20) is a three-dimensional integral. The quantities P^a and L^{ab} are separately invariant under general coordinate transformations of the three-dimensional space and under time reparametrizations, which is an expected feature, since these definitions arise in the Hamiltonian formulation of the theory. Moreover, these quantities transform covariantly under global SO(3,1) transformations [59].

4 Tetrad fields with spherical symmetry

Now we will consider two simple configurations of tetrad fields and discuss their physical interpretation as reference frames. The first one in a quasi-orthogonal coordinate system can be written as [81]

$$\begin{aligned} &e_{(0)}{}^0 = A \,, \quad e_\alpha{}^0 = C x^a \,, \quad e_{(0)}{}^\alpha = D x^\alpha \,, \\ &e_a{}^\alpha = \delta_a{}^\alpha B + F x^a x^\alpha + \epsilon_{a\alpha\beta} S x^\beta \,, \end{aligned} \tag{21}$$

where A, C, D, B, F, and S are unknown functions of r. It can be shown that the unknown functions D and F can be eliminated by coordinate transformations [82, 83], i.e., by making use of the freedom to redefine t and r, leaving the tetrad field (21) having four unknown functions in the quasi-orthogonal coordinates. Thus the tetrad field (21) without the unknown functions D and F and also without the two unknown functions C and S will be used in the following discussion for the calculations of energy, momentum and angular-momentum but in spherical coordinates.

Therefore, the tetrad field (21) can be written in spherical coordinates without the unknown functions D, F, C and S as [83]

$$\left(e_{1a}{}^{\mu}\right) = \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & B\sin\theta\cos\phi & \frac{\cos\theta\cos\phi}{R(r)} & -\frac{\sin\phi}{R(r)\sin\theta} \\ 0 & B\sin\theta\sin\phi & \frac{\cos\theta\sin\phi}{R(r)} & \frac{\cos\phi}{R(r)\sin\theta} \\ 0 & B\cos\theta & -\frac{\sin\theta}{R(r)} & 0 \end{pmatrix}. \quad (22)$$

The other configuration of a tetrad field that has a simple interpretation as a reference frame has the form

$$\left(e_{2a}{}^{\mu}\right) = \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & \frac{1}{R(r)} & 0 \\ 0 & 0 & 0 & \frac{1}{R(r)\sin\theta} \end{pmatrix}. \quad (23)$$

The two tetrads (22) and (23) are related by a local Lorentz transformation that keeps spherical symmetry, i.e., the tetrad (22) can be written in terms of the tetrad (23), using the following local Lorentz transformation:

$$\left(e_{1a}{}^{\mu}\right) = \Lambda_{\nu}{}^{\mu}\left(e_{2a}{}^{\nu}\right), \quad (24)$$

where

$$\Lambda_{\nu}{}^{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}.$$

The space-times associated with the two tetrad fields (22) and (23) are the same and have the form

$$\mathrm{d}s^2 = -A^2\mathrm{d}t^2 + \frac{1}{B^2}\mathrm{d}r^2 + R(r)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \quad (25)$$

Now we are going to calculate the energy, momentum and angular-momentum associated with the two tetrad fields (22) and (23). For asymptotically flat space-times, P^0 yields the ADM energy [84]. In the context of tetrad theories of gravity, asymptotically flat space-times may be characterized by the asymptotic boundary condition

$$e_{a\mu} \cong \eta_{a\mu} + \frac{1}{2}h_{a\mu}(1/r), \quad (26)$$

and by the condition $\partial_\mu e^a{}_\mu = O(1/r^2)$ in the asymptotic limit $r \to \infty$. An important property of tetrad fields that satisfy (26) is that in the flat space-time limit one has $e^a{}_\mu(t,x,y,z) = \delta^a{}_\mu$, and therefore the torsion tensor $T^a{}_{\mu\nu} = 0$.

Now we are going to apply (13) to the tetrad field (22) to calculate the energy content. We perform the calculations in spherical coordinates. Equations (22) and (23) assumed that the reference space is determined by a set of tetrad fields $e^a{}_\mu$ for the flat space-time such that the condition $T^a{}_{\mu\nu} = 0$ is satisfied. Using (5) in (22), the non-vanishing components of the torsion tensor are given by

$$T^{(0)}{}_{01} = \frac{A'}{A}, \quad T^{(2)}{}_{12} = \frac{(1 - R'(r)B)}{R(r)B} = T^{(3)}{}_{13}, \quad (27)$$

and the non-vanishing component of the tensor $T^{(a)}$ defined by (5) is given by

$$T^{(1)} = \frac{B(r)\{2A(r) - 2R'(r)A(r)B(r) - A'(r)B(r)R(r)\}}{R(r)A}. \quad (28)$$

The axial vector associated with (22) is vanishing identically due to the fact that the tetrad field of (22) has a spherical symmetry [81].

Now we are going to apply (13) to the tetrad field (22) using (27) and (28) to calculate the energy content. We perform the calculations in spherical coordinates. The only required component of $\Sigma^{\mu\nu\lambda}$ is

$$\Sigma^{(0)01} = -\frac{R(r)\sin\theta\{1 - R'(r)B\}}{4\pi}. \quad (29)$$

Using (29) in (13) we obtain

$$P^{(0)} = E = -\oint_{S\to\infty} \mathrm{d}S_k \Pi^{(0)k} = -\frac{1}{4\pi}\oint_{S\to\infty} \mathrm{d}S_k e\Sigma^{(0)0k} = R(r)\{1 - R'(r)B\}. \quad (30)$$

Now let us apply (13) to the evaluation of the irreducible mass by fixing V to be the volume within the $r = r_+$ surface, where r_+ is the external horizon, i.e., $B = 0$. Therefore,

$$P^{(0)} = E = -\int_{S_i} \mathrm{d}S_i \Pi^{(0)i} = -\int_S \mathrm{d}\theta\,\mathrm{d}\phi\,\Pi^{(0)1}(r,\theta,\phi), \quad (31)$$

where the surface S is determined by the condition $r = r_+$. The expression of $\Pi^{(0)1}$ will be obtained by considering (14) using (4) and (5). The expression of $\Pi^{(0)1}(r,\theta,\phi)$ for the tetrad (22) reads

$$\Pi^{(0)1}(r,\theta,\phi) = \frac{\sin\theta R(r_+)\{1 - R'(r_+)B(r_+)\}}{4\pi}, \quad (32)$$

integrating (32) on the surface of constant radius $r = r_+$, where r_+ is the external horizon of the black hole. On this surface the second term of (32) vanishes, i.e., $B(r_+) = 0$. Therefore, on the surface $r = r_+$ we get

$$P^{(0)} = E = R(r_+). \quad (33)$$

Equation (33) is consistent with the results obtained before when $R(r_+) = r_+$ [45–50, 66]; otherwise we obtain a different result. It has been shown [4] that the unknown functions in the metric given by (25) may have the value

$$A = \frac{1}{B} = \sqrt{1 - \frac{2M}{r}}, \quad R(r) = r\sqrt{1 - \frac{Q^2 e^{-2\xi_0}}{rM}}. \quad (34)$$

For the value of the unknown functions given by (34) to satisfy the field equation (8) the dilaton, the vector potential, the Maxwell field and the energy-momentum tensor must have the form

$$\begin{aligned}
&\xi=\xi_0-\frac{1}{2}\ln\left(1-\frac{Q^2e^{-2\xi_0}}{rM}\right),\\
&A_3=Q\cos\theta\,,\quad F_{23}=Q\sin\theta\,,\\
&T^0{}_0=\frac{Q^2(4rM^2e^{2\xi_0}-6MQ^2+rQ^2)}{4r^3(rMe^{2\xi_0}-Q^2)^2}\,,\\
&T^1{}_1=\frac{Q^2(4rM^2e^{2\xi_0}-2MQ^2-rQ^2)}{4r^3(rMe^{2\xi_0}-Q^2)^2}\,,\\
&T^2{}_2=T^3{}_3=\\
&-\frac{Q^2(4r^2M^3e^{4\xi_0}-6rM^2Q^2e^{2\xi_0}+2MQ^4-r^2MQ^2e^{2\xi_0}+rQ^4)}{4r^3(rMe^{2\xi_0}-Q^2)^3}\,,
\end{aligned}\qquad(35)$$

where M, Q, ξ and ξ_0 are the mass, charge, dilaton and the asymptotic value of the dilaton respectively. As is clear from (33), if the charge $Q=0$, then $R(r)=r$ and the irreducible mass will coincide with that obtained before [45–50, 66]. Equation (33) tells us that the energy associated with the solution given by (34) on the surface $R(r_+)$ is different from what is well known [45–50, 66]. To overcome such a problem let us use a coordinate transformation that makes $R(r)$ in (34) appear to be r.

Now we are going to redefine the radial coordinate to be

$$r=\sqrt{R^2+\frac{Q^4e^{-4\xi_0}}{4M^2}}+\frac{Q^2e^{-2\xi_0}}{2M}\,.\qquad(36)$$

Using the coordinate transformation (36) in (22) we get

$$(e_{1a}{}^{\mu})=\begin{pmatrix}
\frac{1}{\sqrt{1-\frac{4M^2e^{2\xi}}{\lambda(R)}}} & 0 & 0 & 0\\
0 & \frac{\sigma(R)\sqrt{1-\frac{4M^2e^{2\xi}}{\lambda(R)}}\sin\theta\cos\phi}{2MR} & \frac{\cos\theta\cos\phi}{R} & -\frac{\sin\phi}{R\sin\theta}\\
0 & \frac{\sigma(R)\sqrt{1-\frac{4M^2e^{2\xi}}{\lambda(R)}}\sin\theta\sin\phi}{2MR} & \frac{\cos\theta\sin\phi}{R} & \frac{\cos\phi}{R\sin\theta}\\
0 & \frac{\sigma(R)\sqrt{1-\frac{4M^2e^{2\xi}}{\lambda(R)}}\cos\theta}{2MR} & -\frac{\sin\theta}{R} & 0
\end{pmatrix},\qquad(37)$$

where

$$\begin{aligned}
\lambda(R)&\stackrel{\text{def.}}{=}\sqrt{4R^2M^2+Q^4e^{-4\xi_0}}\,e^{2\xi_0}+Q^2\,,\\
\sigma(R)&\stackrel{\text{def.}}{=}\sqrt{4R^2M^2+Q^4e^{-4\xi_0}}\,,
\end{aligned}\qquad(38)$$

and the associated spacetime of (37) is given by

$$\begin{aligned}
\mathrm{d}s^2=&-\left(1-\frac{4M^2e^{2\xi}}{\lambda(R)}\right)\mathrm{d}t^2+\frac{4M^2R^2}{\sigma^2(R)\left(1-\frac{4M^2e^{2\xi}}{\lambda(R)}\right)}\mathrm{d}R^2\\
&+R^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\,.
\end{aligned}\qquad(39)$$

When the dilaton ξ and the charge Q are vanishing, (37) will be identical with the tetrad field that reproduces the Schwarzschild spacetime [85]. When the dilation solution is vanishing, (37) will behave asymptotically like

$$(e_{1a}{}^{\mu})\cong\begin{pmatrix}
\frac{2R^2+2MR+3M^2-Q^2}{2R^2} & 0 & 0 & 0\\
0 & \frac{\sin\theta\cos\phi(2R^2-2MR-M^2+Q^2)}{2R^2} & \frac{\cos\theta\cos\phi}{R} & -\frac{\sin\phi}{R\sin\theta}\\
0 & \frac{\sin\theta\sin\phi(2R^2-2MR-M^2+Q^2)}{2R^2} & \frac{\cos\theta\sin\phi}{R} & \frac{\cos\phi}{R\sin\theta}\\
0 & \frac{\cos\theta(2R^2-2MR-M^2+Q^2)}{2R^2} & -\frac{\sin\theta}{R} & 0
\end{pmatrix},\qquad(40)$$

and the associated metric of (40) has the form

$$\begin{aligned}
\mathrm{d}s^2\cong&-\left(1-\frac{2MR-Q^2}{R^2}\right)\mathrm{d}t^2+\left(1+\frac{2MR-Q^2}{R^2}\right)\mathrm{d}R^2\\
&+R^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\,,
\end{aligned}\qquad(41)$$

which is the asymptotic form of Reissner–Nordström metric [85].

Now one should repeat the calculations of energy using the tetrad field given by (37). Using (5) in (37), the non-vanishing components of the torsion tensor are given by

$$\begin{aligned}
T^0{}_{01}&=\frac{2\lambda'(R)M^2e^{2\xi_0}}{\lambda(R)[\lambda(R)-4M^2e^{\xi_0}]}\,,\\
T^2{}_{12}&=\frac{\sigma(R)\sqrt{\lambda(R)-4M^2e^{2\xi_0}}-2MR\sqrt{\lambda(R)}}{R\sigma(R)\sqrt{\lambda(R)-4M^2e^{2\xi_0}}}=T^3{}_{13}\,,
\end{aligned}\qquad(42)$$

and the non-vanishing component of the tensor $T^{(a)}$ is given by

$$T^{(1)}=\frac{\begin{array}{c}-\sigma(R)\{\sigma(R)M^2e^{2\xi_0}(4\lambda(R)-R\lambda'(R))\\+2RM\sqrt{\lambda^4(R)-4\lambda^3(R)M^2e^{2\xi_0}}-\lambda^2(R)\sigma(R)\}\end{array}}{2R^3M^2\lambda^2(R)}\,.\qquad(43)$$

Using (42) and (43) one calculates the energy content. The only required component of $\Sigma^{\mu\nu\lambda}$ is

$$\Sigma^{(0)01}=-\frac{\sin\theta\{2MR\sqrt{\lambda(R)}-\sigma(R)\sqrt{\lambda(R)-4M^2e^{2\xi_0}}\}}{8M\pi\sqrt{\lambda(R)}}\,.\qquad(44)$$

Substituting (44) in (13) we obtain

$$P^{(0)}=E=-\oint_{S\to\infty}\mathrm{d}S_k\Pi^{(0)k}=-\frac{1}{4\pi}\oint_{S\to\infty}\mathrm{d}S_k e\Sigma^{(0)0k}$$
$$=\frac{\{2MR\sqrt{\lambda(R)}-\sigma(R)\sqrt{\lambda(R)-4M^2e^{2\xi_0}}\}}{2M\sqrt{\lambda(R)}}$$
$$=R-\frac{\left[\begin{array}{l}e^{-2\xi_0}\Big(\sqrt{4R^2M^2e^{4\xi_0}+Q^4}\\ \times\sqrt{\sqrt{4M^2R^2e^{4\xi_0}+Q^4}+Q^2-4M^2e^{2\xi_0}}\Big)\end{array}\right]}{2M\sqrt{\sqrt{4M^2R^2e^{4\xi_0}+Q^4}+Q^2}},$$
$$\cong M-\frac{4Q^2M^2e^{-2\xi_0}-4M^4+Q^4e^{-4\xi_0}}{8M^2R}+O\left(\frac{1}{R^2}\right), \tag{45}$$

where we have used the definitions of $\lambda(R)$ and $\sigma(R)$ given by (38). For large R, i.e., $\lim_{R\to\infty}$, (45) will give the ADM [86]. If the asymptotic dilaton ξ_0 is vanishing, then the asymptotic form of the energy can be obtained from (45) to have the value

$$E\cong M-\frac{4Q^2M^2-4M^4+Q^4}{8M^2R}, \tag{46}$$

which is the energy of the Reissner–Nordström space-time when $Q^4=0$ and $M^2=0$ [85].

Now we may apply (13) to the evaluation of the irreducible mass by fixing V to be the volume within the $R=R_+$ surface where R_+ is the external horizon, i.e., $\left(1-\frac{4M^2e^{2\xi}}{\lambda(R_+)}\right)\sigma^2(R_+)=0$. Therefore,

$$P^{(0)}=E=-\int_{S_i}\mathrm{d}S_i\Pi^{(0)i}(R,\theta,\phi)$$
$$=-\int_S\mathrm{d}\theta\,\mathrm{d}\phi\,\Pi^{(0)1}(R,\theta,\phi), \tag{47}$$

where the surface S is determined by the condition $R=R_+$. The expression of $\Pi^{(0)1}$ will be obtained by considering (14) using (4) and (5). The expression of $\Pi^{(0)1}(R,\theta,\phi)$ for the tetrad (37) reads

$$\Pi^{(0)1}(R,\theta,\phi)$$
$$=\frac{\{2MR_+\sqrt{\lambda(R_+)}-\sigma(R_+)\sqrt{\lambda(R_+)-4M^2e^{2\xi_0}}\}}{8M\pi\sqrt{\lambda(R_+)}}, \tag{48}$$

where R_+ is the external horizon of the black hole. On this surface the second term of (48) vanishes, i.e., $\sqrt{\lambda(R_+)-4M^2e^{2\xi_0}}\sigma(R_+)$. Therefore, on the surface $R=R_+$ integration of (48) will give

$$P^{(0)}=E=R_+, \tag{49}$$

which is a satisfactory result that has been obtained before [45–50, 66].

Using (14) in (37) one calculates the momentum and angular-momentum associated with the first tetrad field given by (37). In this case we get

$$\Pi^{(1)1}(R,\theta,\phi)=0. \tag{50}$$

Substituting (50) in (13) we get

$$P^{(1)}=\int_V\mathrm{d}V\partial_1(\Pi^{(1)1}(R,\theta,\phi))$$
$$=\int_S\mathrm{d}S_1\Pi^{(1)1}(R,\theta,\phi)=0. \tag{51}$$

By the same method we obtain

$$\Pi^{(2)1}(R,\theta,\phi)=0,\qquad P^{(2)}=0,$$
$$\Pi^{(3)1}(R,\theta,\phi)=0,\qquad P^{(3)}=0. \tag{52}$$

The results of (51) and (52) are expected results, since the space-time given by (37) is a spherically symmetric static space-time. Therefore, the spatial momentum associated with any static solution is identically vanishing [86].

We use (19) and (4) in (20) to calculate the components of the angular-momentum. Finally we get

$$M^{(0)(1)}(R,\theta,\phi)=\frac{\begin{array}{r}-R\sin^2\theta\cos\phi\{2MR\sqrt{\lambda^2(R)-4M^2\lambda(R)e^{2\xi_0}}\\ +\sigma(R)\lambda(R)-4\sigma(R)M^2e^{2\xi_0}\}\end{array}}{4\pi\sigma(R)\lambda(R)},$$
$$M^{(0)(2)}(R,\theta,\phi)=\frac{\begin{array}{r}-R\sin^2\theta\sin\phi\{2MR\sqrt{\lambda^2(R)-4M^2\lambda(R)e^{2\xi_0}}\\ +\sigma(R)\lambda(R)-4\sigma(R)M^2e^{2\xi_0}\}\end{array}}{4\pi\sigma(R)\lambda(R)},$$
$$M^{(0)(3)}(R,\theta,\phi)=\frac{\begin{array}{r}-R\sin\theta\cos\theta\{2MR\sqrt{\lambda^2(R)-4M^2\lambda(R)e^{2\xi_0}}\\ +\sigma(R)\lambda(R)-4\sigma(R)M^2e^{2\xi_0}\}\end{array}}{4\pi\sigma(R)\lambda(R)},$$
$$M^{(1)(2)}(R,\theta,\phi)=M^{(1)(3)}(R,\theta,\phi)=M^{(2)(3)}(R,\theta,\phi)=0. \tag{53}$$

Using (53) in (20) we get

$$L^{(0)(1)}=\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}R M^{(0)(1)}(R,\theta,\phi)=0,$$

and by the same method we get

$$L^{(0)(2)}=L^{(0)(3)}=L^{(1)(2)}=L^{(1)(3)}=L^{(2)(3)}=0. \tag{54}$$

It is of interest to note that the vanishing of $L^{(0)(1)}$ and $L^{(0)(2)}$ is due to the appearance of terms like $\sin\phi$ and $\cos\phi$, while the vanishing of $L^{(0)(3)}$ is due to the appearance of terms like $\sin\theta\cos\theta$.

To repeat the same computation for the tetrad (23) it is sufficient to use the transformation rules of the conserved quantities [87]. Therefore, the required component of $\Sigma^{\mu\nu\lambda}$

needed to calculate the energy of the tetrad field (23) has the form

$$\Sigma^{(0)01} = -\frac{B(r)R(r)R'(r)\sin\theta}{4\pi}\,. \tag{55}$$

Substituting (55) in (13) we obtain

$$\begin{aligned} P^{(0)} = E &= -\oint_{S\to\infty} \mathrm{d}S_k \Pi^{(0)k}(r,\theta,\phi) \\ &= -\frac{1}{4\pi}\oint_{S\to\infty} \mathrm{d}S_k e\Sigma^{(0)0k} = -B(r)R(r)R'(r) \\ &= -\frac{(2rM - Q^2 e^{-2\xi_0})\sqrt{1-\frac{2M}{r}}}{2M}\,. \end{aligned} \tag{56}$$

When the asymptotic dilaton obeys $\xi_0 = 0$ and the charge $Q = 0$, the asymptotic form of the above form of energy is given by

$$E \cong M - r\,, \tag{57}$$

which is different from the ADM form [86]. This is due to the fact that the components of the torsion when $M = 0$, $Q = 0$ and $\xi_0 = 0$ do not in the vanishing case identically contradict the flatness condition given by (26). Therefore, in this case we are going to use the regularized expression for the gravitational energy-momentum [45–50].

5 Regularized expression for the gravitational energy-momentum and localization of energy

An important property of the tetrad fields that satisfies the condition of (26) is that in the flat space-time limit ${e^a}_\mu(t,x,y,z) = {\delta^a}_\mu$, and therefore the torsion ${T^\lambda}_{\mu\nu} = 0$. Hence for the flat space-time it is usual to consider a set of tetrad fields such that ${T^\lambda}_{\mu\nu} = 0$ *in any coordinate system*. However, in general an arbitrary set of tetrad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by (26). Moreover, for such tetrad fields the torsion obeys ${T^\lambda}_{\mu\nu} \neq 0$ for the flat space-time [88]. It might be argued, therefore, that the expression for the gravitational energy-momentum (13) is restricted to particular class of tetrad fields, namely, to the class of frames such that ${T^\lambda}_{\mu\nu} = 0$ if ${E_a}^\mu$ represents the flat space-time tetrad field [88]. To explain this, let us calculate the flat space-time of the tetrad field of (23) using (34), which is given by

$$\left({E_{2a}}^\mu\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}. \tag{58}$$

Equation (58) yields the following non-vanishing torsion components:

$$T^{(2)}{}_{12} = -\frac{1}{r} = T^{(3)}{}_{13}\,, \quad T^{(3)}{}_{23} = -\cot\theta\,. \tag{59}$$

The tetrad field (58) when written in Cartesian coordinates will have the form

$$\left({E_{2a}}^\mu(t,x,y,z)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ 0 & \frac{xz}{r\sqrt{x^2+y^2}} & \frac{yz}{r\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r} \\ 0 & -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{pmatrix}. \tag{60}$$

In view of the geometric structure of (60), we see that (23) does not display the asymptotic behavior required by (26). Moreover, in general the tetrad field (60) is adapted to accelerated observers [43, 45–50, 88]. To explain this, let us consider a boost in the x-direction of (60). We find

$$\begin{aligned} &\left({E_{2a}}^\mu(t,x,y,z)\right) \\ &= \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ \frac{v\gamma x}{r} & \frac{\gamma x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{v\gamma xz}{r\sqrt{x^2+y^2}} & \frac{\gamma xz}{r\sqrt{x^2+y^2}} & \frac{yz}{r\sqrt{x^2+y^2}} & \frac{-\sqrt{x^2+y^2}}{r} \\ \frac{-v\gamma y}{\sqrt{x^2+y^2}} & \frac{-\gamma y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{pmatrix}, \end{aligned} \tag{61}$$

where v is the speed of the observer and $\gamma = \frac{1}{\sqrt{1-v^2}}$. For a static object in a space-time whose four-velocity is given by $u^\mu = (1,0,0,0)$ we may compute its frame components $u^a = {e^a}_\mu u^\mu = \left(\gamma, \frac{v\gamma x}{r}, \frac{v\gamma xz}{r\sqrt{x^2+y^2}}, \frac{-v\gamma y}{\sqrt{x^2+y^2}}\right)$. It can be shown that along an observer's trajectory whose velocity is determined by u^a the quantities

$$\phi_{(j)}^{(k)} = u^i\left({E_2}^{(k)}{}_m \partial_i {E_{2(j)}}^m\right), \tag{62}$$

constructed from (61), are non-vanishing. This fact indicates that along the observer's path the spatial axis ${E_{2(a)}}^\mu$ rotates [43, 88]. In spite of the above problems discussed for the tetrad field of (23) it yields a satisfactory value for the total gravitational energy-momentum, as we will discuss.

In (13) it is implicitly assumed that the reference space is determined by a set of tetrad fields ${e^a}_\mu$ for flat space-time, such that the condition ${T^a}_{\mu\nu} = 0$ is satisfied. However, in general there exist flat space-time tetrad fields for which ${T^a}_{\mu\nu} \neq 0$. In this case (13) may be generalized [43, 88] by adding a suitable reference space subtraction term, exactly like in the Brown–York formalism [89–91].

We will denote ${T^a}_{\mu\nu}(E) = \partial_\mu {e^a}_\nu - \partial_\nu {e^a}_\mu$ and $\Pi^{aj}(E)$ as the expression of Π^{aj} constructed by the flat tetrad ${E^a}_\mu$. *The regularized form of the gravitational energy-momentum* P^a *is defined by* [43, 88]

$$P^a = -\int_V \mathrm{d}^3x \partial_k [\Pi^{ak}(e) - \Pi^{ak}(E)]\,. \tag{63}$$

This condition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by tetrad fields ${E^a}_\mu$, obtained from ${e^a}_\mu$ by requiring the vanishing of the physical parameters like mass, angular-momentum, etc. Assuming that the space-time is asymptotically flat, (63) may have the form [43, 88]

$$P^a = -\oint_{S\to\infty} \mathrm{d}S_k [\Pi^{ak}(e) - \Pi^{ak}(E)]\,, \tag{64}$$

where the surface S is established at spacelike infinity. Equation (64) transforms as a vector under the global SO(3,1) group [45–50].

We may likewise establish the regularized expression for the gravitational four-angular-momentum. It reads

$$L^{ab} = \int_V \mathrm{d}^3x [M^{ab}(e) - M^{ab}(E)]\,. \tag{65}$$

Now we are in a position to proof that the tetrad field (23) yields a satisfactory value for the total gravitational energy-momentum. We will integrate (64) over a surface of constant radius $x^1 = r$ and require $r \to \infty$. Therefore, the index k in (64) takes the value $k = 1$. We need to calculate the quantity

$$\Sigma^{(0)01} = e^{(0)}{}_0 \Sigma^{001} = \frac{1}{2} e^{(0)}{}_0 (T^{001} - g^{00} T^1)\,.$$

Evaluating the above equation we find

$$\Pi^{(0)1}(e) = \frac{-1}{4\pi} e \Sigma^{(0)01} = -\frac{\sin\theta (2rM - Q^2 e^{-2\xi_0})\sqrt{1-\frac{2M}{r}}}{8\pi M}\,, \tag{66}$$

and the expression of $\Pi^{(0)1}(E)$ is obtained by just making $M = 0$, $Q = 0$ and $\xi_0 = 0$ in (66); it is given by

$$\Pi^{(0)1}(E) = \frac{-1}{4\pi} r \sin\theta\,. \tag{67}$$

Thus the gravitational energy of the tetrad field of (23) is given by

$$P^{(0)} = \int \mathrm{d}\theta\, \mathrm{d}\phi \frac{1}{4\pi} \sin\theta \left(r - \frac{(2rM - Q^2 e^{-2\xi_0})\sqrt{1-\frac{2M}{r}}}{2M} \right),$$
$$r - \frac{(2rM - Q^2 e^{-2\xi_0})\sqrt{1-\frac{2M}{r}}}{2M} \cong M + \frac{Q^2 e^{-2\phi_0}}{2M} + O\left(\frac{1}{r}\right), \tag{68}$$

which is exactly the ADM when $Q^2 = 0$ up to $O(1/r)$. Equation (68) tells us that when $(2rM - Q^2 e^{-2\xi_0}) = 0$ the form of the energy given by (68) will follow and this is one of the defects of the solution given by (34) [4]. Therefore, we use the coordinate transformation given by (36). The tetrad (23) after using the transformation (36) will have the form

$$\left(e_{2a}{}^{\mu}\right) = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{4M^2 e^{2\xi}}{\lambda(R)}}} & 0 & 0 & 0 \\ 0 & \frac{\sigma(R)\sqrt{1-\frac{4M^2 e^{2\xi}}{\lambda(R)}}}{2MR} & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{R\sin\theta} \end{pmatrix}. \tag{69}$$

Repeating the calculations above, the non-vanishing components of the torsion tensor and the vector field $T^{(a)}$ of the tetrad field given by (69) have the form

$$T^{(0)}{}_{01} = -\frac{2M^2\lambda'(R) e^{2\xi_0}}{\lambda(R)[\lambda(R) - 4M^2 e^{2\xi_0}]}\,,$$
$$T^{(2)}{}_{12} = T^{(3)}_{13} = -\frac{1}{R}\,, \quad T^{(3)}{}_{23} = -\cot\theta\,,$$
$$T^{(1)} = -\frac{\sigma(R)\{\lambda'(R) - 4\lambda(R)M^2 e^{2\xi_0} - RM^2\lambda'(R) e^{2\xi_0}\}}{R^3 M^2 \lambda^2(R)}\,,$$
$$T^{(2)} = -\frac{\cot\theta}{R^2}\,. \tag{70}$$

The only required component of $\Sigma^{\mu\nu\lambda}$ needed to calculate the energy using the regularized expression given by (64) is

$$\Sigma^{(0)01}(e) = \frac{\left(\frac{\sigma(R)\sqrt{\lambda(R) - 4M^2 e^{2\xi_0}}}{2M\sqrt{\lambda(R)}}\right)\sin\theta}{4\pi}\,,$$
$$\Sigma^{(0)01}(E) = \frac{R\sin\theta}{4\pi}\,. \tag{71}$$

Substituting (71) in (64) we obtain

$$P^{(0)} = E = -\oint_{S\to\infty} \mathrm{d}S_k \Pi^{(0)k}(R,\theta,\phi)$$
$$= -\frac{1}{4\pi}\oint_{S\to\infty} \mathrm{d}S_k e \Sigma^{(0)0k}$$
$$= R - \frac{\sigma(R)\sqrt{\lambda(R) - 4M^2 e^{2\xi_0}}}{2M\sqrt{\lambda(R)}}$$
$$= R - \frac{\left[\sqrt{4M^2R^2 + Q^4 e^{-4\xi_0}} \times \sqrt{\sqrt{4M^2R^2 + Q^4 e^{-4\xi_0}}\, e^{2\xi_0} + Q^2 - 4M^2 e^{2\xi_0}}\right]}{2M\sqrt{\sqrt{4M^2R^2 + Q^4 e^{-4\xi_0}}\, e^{2\xi_0} + Q^2}}$$
$$\cong M + O\left(\frac{1}{R}\right),$$

which is the ADM up to $O\left(\frac{1}{R}\right)$,

$$\cong M - \frac{4Q^2M^2 e^{-2\xi_0} + 4M^4 - Q^4 e^{-4\xi_0}}{8M^2 R} + O\left(\frac{1}{R^2}\right), \tag{72}$$

which is the energy of Reissner–Nordström space-time when the asymptotic dilaton $\xi_0 = 0$, $Q^4 = 0$ and $M^2 = 0$ up to $O(\frac{1}{R^2})$ [85].

By the same method used for the first tetrad given by (37) we find that the momentum and angular-momentum associated with the second tetrad field given by (69) are

$$\Pi^{(1)1}(R,\theta,\phi) = 0\,,$$
$$P^{(1)} = \int_V \mathrm{d}V \partial_1 (\Pi^{(1)1}(R,\theta,\phi)) = \int_S \mathrm{d}S_1 \Pi^{(1)1}(R,\theta,\phi) = 0\,,$$
$$\Pi^{(2)1}(R,\theta,\phi) = 0\,, \quad P^{(2)} = 0\,,$$
$$\Pi^{(3)1}(R,\theta,\phi) = 0, \quad P^{(3)} = 0\,. \tag{73}$$

The non-vanishing components of the angular-momentum are given by

$$M^{(0)(1)}(e) = \frac{R\sin\theta(\lambda(R) - 4M^2 e^{2\xi_0})}{4\pi\lambda(R)} \cong \frac{\sin\theta(R-M)}{4\pi} + O\left(\frac{1}{R}\right),$$
$$M^{(0)(1)}(E) \cong \frac{R\sin\theta}{4\pi} + O\left(\frac{1}{R}\right),$$
$$M^{(0)(2)}(R,\theta,\phi) = \frac{MR^2\cos\theta\sqrt{(\lambda(R)-4M^2e^{2\xi_0})}}{4\pi\sigma(R)\sqrt{\lambda(R)}},$$
$$M^{(0)(3)}(R,\theta,\phi) = M^{(1)(2)}(R,\theta,\phi) = M^{(1)(3)}(R,\theta,\phi) = M^{(2)(3)}(R,\theta,\phi) = 0\,. \quad (74)$$

Using (74) in (65) we get

$$L^{(0)(1)} = \int_0^\pi \int_0^{2\pi} \int_0^\infty \mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}R[M^{(0)(1)}(e) - M^{(0)(1)}(E)] = M\int_0^\infty \mathrm{d}R\,, \quad (75)$$

which give an infinite result! By the same method we can obtain

$$L^{(0)(2)} = L^{(0)(3)} = L^{(1)(2)} = L^{(1)(3)} = L^{(2)(3)} = 0\,. \quad (76)$$

It is of interest to note that the non-vanishing of $L^{(0)(1)}$ is due to the appearance of terms like $\sin\theta$, while the vanishing of $L^{(0)(2)}$ is due to the appearance of terms like $\cos\theta$.

We show by explicit calculation that the energy-momentum tensor, which is coordinate independent, does not give a consistent result of the angular-momentum when applied to the tetrad field given by (23), which does not satisfy the boundary condition given by (26).

6 Main results and discussion

The main results of this paper are the following.

- Two different tetrad fields are used. The space-time associated with these tetrad fields is given by (25).
- The energy of these tetrad fields is calculated using the gravitational energy-momentum tensor, which is coordinate independent [45–50]. One of these tetrad fields, given by (22), gives a satisfactory result for the energy after using the coordinate transformation given by (36). The other tetrad field that is given by (23), its associated energy, depends on the radial coordinate.
- Calculations of the torsion components associated with the two tetrad fields are given. From these calculations we show that the torsion components of each tetrad field are different. This may give an indication of why the energy of the two tetrad fields is different.
- We use the regularized expression of the gravitational energy-momentum tensor to calculate the energy associated with the second tetrad field given by (23).
- We have shown that the energy associated with the second tetrad field did not give a consistent result even after using the regularized expression of the gravitational energy-momentum tensor. Therefore, we use the coordinate transformation given by (36). Applying this coordinate transformation to the tetrad field (23) we have got a satisfactory value of energy that coincides with the value of energy of the first tetrad field.
- Using the definition of the energy and the angular-momentum given by (13) and (20) we show by explicit calculations that the angular-momentum depends on the choice of the frame used.
- The calculation of the irreducible mass is given within the external horizons using the Hamiltonian formulation. From this calculation we show that the external horizons of each model do not play any role for the energy.
- We have shown by explicit calculations that the diagonal tetrad field, which is given by (23), suffers from some problems.
  - i) It does not satisfy the condition given by (26), which guarantees the flatness of spacetime; consequently, the components of the torsion tensor did not vanish when the physical quantities are set equal to zero.
  - ii) The use of the energy-momentum tensor given by (13) did not give consistent results! Therefore, we have used the regularized expression of the energy-momentum tensor and got a consistent result for the energy. Also we have shown that (20) and (65) gave infinite results on calculating the angular-momentum [86]!
- The construction of the tetrad given by (23) is the square root of the metric given by (25); meanwhile, the construction of the tetrad given by (22) is not the square root of (25). A possible interpretation of the result given by the second tetrad (which is the square root of the metric) is that *it may not be a physical one*. The same problem has appeared [88] for the Kerr solution. We need more studies to confirm this conclusion.

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